THE UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

NON-LINEAR DIFFERENTIAL EQUATIONS HAVING BOTH QUADRATIC DAMPING AND STIFFNESS

by

A.W. Babister, M.A., Ph.D.

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SUMMARY

The nature of solutions of the autonomous equation

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0$$

is considered. The trajectories in the $(x, \dot{x})$ phase plane are given for all combinations of sign of b_0, μ, c_1 and c_2. It is shown that self-sustaining oscillations occur if $b_0 = 0$ and $c_1 > 0$ for all values of μ and c_2 (for sufficiently small amplitudes in x).

The stability of systems satisfying this differential equation is discussed, together with methods of improving the character of the vibrations.

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General Introduction

This report is one of a series on dynamical systems with non-linear characteristics (Babister, 1972 and 1973). In an earlier report we considered the nature of solutions of the autonomous equation

$$\ddot{x} + (b_0 + b_1 x)\dot{x} + c_1 x + c_2 x^2 = 0, \tag{1}$$

We now study solutions (and, in particular, phase-plane trajectories) of the autonomous equation

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0 \tag{2}$$

where b_0, c_1, c_2 and μ are constants. Systems satisfying equations (1) or (2) can be thought of as being simple perturbations of the linear system

$$\ddot{x} + b_0\dot{x} + c_1 x = 0. \tag{3}$$

However, in both (1) and (2), the stiffness terms $(c_1 x + c_2 x^2)$ are quadratic in the displacement x, and the system (2) also has quadratic damping $(b_0\dot{x} + \mu\dot{x}^2)$.

As in the previous reports we shall be dealing mainly with local variations in the neighbourhood of singular points of (2), the aim being to determine conditions under which the system (2) has solutions which are similar to those of the linear system (3). We shall also consider the global variation of the set of trajectories of (2) in the (x,y) phase plane, where

$$\dot{x} = y. \tag{4}$$

From (2) and (4),

$$\dot{y} = -b_0 y - \mu y^2 - c_1 x - c_2 x^2. \tag{5}$$

We note that $\dot{x}$ and $\dot{y}$ are, in general, simultaneously zero only at 0.

From (4) and (5),

$$\frac{dy}{dx} = -b_o - \mu y - \frac{g(x)}{y} \tag{6}$$

where $g(x) = c_1 x + c_2 x^2$ (7)

From (6) we see that $\frac{dy}{dx}$ will be infinite where $y = 0$ (unless $g(x) = 0$ at the same point in the phase plane).

To find possible slopes of trajectories passing through 0, put $y = \lambda x$ in (6). On letting $x \to 0$, with $g(0) = 0$, we obtain

$$\lambda = -b_o - \frac{g'(0)}{\lambda}$$

that is, using (7),

$$\lambda^2 + b_o\lambda + c_1 = 0.$$

Thus, for the systems we are considering, the local slopes of the trajectories passing through 0 satisfy the same characteristic equation as for the system obtained by retaining only the linear terms in (2).

There are a number of general theorems linking the properties of the system (2) with those of the linear system (3) (see Sansone and Conti, 1964, and Loud, 1964). If $0 < b_o^2 < 4c_1$, 0 is a focus for the system (3) and it is also a focus for (2). If $b_o = 0$ and $c_1 > 0$, 0 is a centre for (3); it is also a centre for (2).

The physical characteristics of systems satisfying (2) can also be seen by considering the energy function E, defined by

$$E = \tfrac{1}{2}\dot{x}^2 + \int g(x)\,dx \tag{8}$$

where $g(x)$ is given by (7). Differentiating (8) with respect to t and using (2) and (4), we obtain

$$\frac{dE}{dt} = -(b_0 + \mu y)y^2 \tag{9}$$

Thus energy will be dissipated with time for the system (2) if $b_0 + \mu y > 0$.

For the system considered,

$$\frac{d^2x}{dt^2} + b_0\frac{dx}{dt} + \mu\left(\frac{dx}{dt}\right)^2 + c_1x + c_2x^2 = 0, \tag{10}$$

the point $x = 0$ is an equilibrium point. For such a system the origin is a singular point in the $(x,\dot{x})$ phase plane. It is also seen that the point $x = -\frac{c_1}{c_2}$ is another equilibrium point for the system. On putting $x = z - \frac{c_1}{c_2}$, we see that (10) becomes

$$\ddot{z} + b_0\dot{z} + \mu\dot{z}^2 - c_1z + c_2z^2 = 0 \tag{11}$$

which is an equation of the same form as (10). More generally, any equation of the form

$$\ddot{z} + b_0\dot{z} + \mu\dot{z}^2 + c_0 + c_1z + c_2z^2 = 0 \tag{12}$$

in which c_1 and c_2 are not <u>both</u> zero, can be put in the form of (10) with real coefficients on letting $x = z - \gamma$, where γ is a real constant provided that the equation

$$c_0 + c_1\gamma + c_2\gamma^2 = 0$$

has a real root, that is, provided that $c_1^{\,2} \geqslant 4c_0c_2$.

In (10), put $x = \alpha X$, $t = \beta T$, where α and β are constants. Then

$$\frac{d^2X}{dT^2} + \beta b_0 \frac{dX}{dT} + \alpha\mu\left(\frac{dX}{dT}\right)^2 + \beta^2 c_1 X + \alpha\beta^2 c_2 X^2 = 0. \tag{13}$$

Thus, if (10) has the solution $x = \phi(t)$, with $x = x_0$, $y = y_0$ at $t = 0$, (13) has the solution $X = \alpha^{-1}\phi(\beta T)$, with $X = \frac{x_0}{\alpha}$ and $\frac{dX}{dT} = \frac{\beta y_0}{\alpha}$ at $T = 0$. In particular we note that a variation in the value of α merely affects the non-linear terms in (13), and that if β is replaced by $-\beta$, t is changed to $-t$. Thus, if (10) has a periodic solution, (13) with any non-zero α and β will also have a periodic solution. If $\alpha = -1$ and $\beta = 1$, the coefficients μ and c_2 in (10) become $-\mu$ and $-c_2$, and the variation of X with T is identical with that of (-x) with t. Again, if $\alpha = 1$ and $\beta = -1$, the coefficient b_0 in (10) becomes $-b_0$, and the variation of X with T is identical with that of x with (-t). Thus the positive semi-trajectory ($T > 0$) in the X plane is the same as the negative semi-trajectory ($t < 0$) in the x plane.

Scaling factors were used in the numerical solutions given in this report, many of which were calculated on Glasgow University's analogue computer (PACE). The computer calculations were carried out for the equation

$$0.1\frac{d^2X}{dT^2} + 0.1b_0\frac{dX}{dT} + 0.2\mu\left(\frac{dX}{dT}\right)^2 + 0.1c_1 X + 0.2c_2 X^2 = 0 \tag{14}$$

with b_0, μ, c_1 and c_2 each having the values 1, 0, -1. Thus the solutions were performed in real time ($\beta = 1$) with a scaling factor $\alpha = 2$.

In part 1 of this report we discuss the nature of solutions of the differential equation (10) with $c_2 = 0$, and in part 2 we deal with the case $c_2 \neq 0$. For ease of reference the various cases considered are set out in table 1.

TABLE 1

Index to discussion of solutions of

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0$$

Para.	Case	μ	c_1	c_2	General remarks
		PART ONE			
1.2	1	0	0	0	
1.2	2	+	0	0	
1.2	3	−	0	0	
1.3	4	0	+	0	
1.3	5	+	+	0	Some periodic solutions
1.3	6	−	+	0	
1.4	7	0	−	0	
1.4	8	+	−	0	
1.4	9	−	−	0	
		PART TWO			
2.2	10	0	0	+	
2.2	11	+	0	+	
2.2	12	−	0	+	
2.3	13	0	0	−	
2.3	14	+	0	−	
2.3	15	−	0	−	
2.4	16	0	+	+	
2.4	17	+	+	+	Some periodic solutions
2.4	18	−	+	+	
2.5	19	0	+	−	
2.5	20	+	+	−	Some periodic solutions
2.5	21	−	+	−	
2.6	22	0	−	+	
2.6	23	+	−	+	Some periodic solutions
2.6	24	−	−	+	
2.6	25	0	−	−	
2.6	26	+	−	−	Some periodic solutions
2.6	27	−	−	−	

Part 1 Non-linear differential equations of the form

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x = 0$$

1.1 Introduction

We consider solutions of the equation

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x = 0 \tag{15}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -b_0 y - \mu y^2 - c_1 x \end{aligned}\right\} \tag{16}$$

where b_0, c_1 and μ are real constants. In particular we shall show how the nature of the solution depends upon the initial conditions $x = x_0$, $y = \dot{x}_0 = y_0$ at time $t = 0$.

1.2 Systems with zero stiffness ($c_1 = 0$)

Case 1 $\quad \mu = 0$, $c_1 = 0$.

$$\ddot{x} + b_0\dot{x} = 0. \tag{17}$$

Equation (17) is a linear differential equation. The general solution and the trajectories in the phase plane were discussed in the previous report (Babister, 1973).

Case 2 $\quad \mu > 0$, $c_1 = 0$.

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 = 0, \tag{18}$$

Equation (18) has a first integral

$$y = \dot{x} = -\frac{b_0}{\mu} + Ae^{-\mu x} \tag{19}$$

where A is a constant.

Now, with $y = \dot{x}$, (18) can be put in the form

$$\frac{1}{y}\frac{dy}{dt} + b_o + \mu\frac{dx}{dt} = 0$$

which can be integrated to give

$$\log|y| + b_o t + \mu x + b_o B = 0 \quad (b_o \neq 0) \tag{20}$$

where B is a constant.
From (19) and (20), we obtain, on eliminating y, with $b_o \neq 0$,

$$b_o(t + B) = -\mu x - \log\left|Ae^{-\mu x} - \frac{b_o}{\mu}\right| \tag{21}$$

If $b_o = 0$, on integrating (19) we obtain

$$(t + B) = \frac{e^{\mu x}}{A\mu} . \tag{22}$$

Trajectories in the (x,y) phase plane are given in fig. 1 (in which y is plotted against μx for $b_o = 0$). If $y_o = 0$, B is infinite and $x = x_o$ for all values of t; the phase plane curve is then a point. Thus there is a whole line of equilibrium points along the x axis. From (22) we see that, if $b_o = 0$, the solution is unbounded as $t \to \infty$. If $b_o > 0$ and $y_o > -\frac{b_o}{\mu}$, the solution is bounded and, from (21), $\mu x \to \log\frac{\mu A}{b_o}$ as $t \to \infty$. If $b_o < 0$ or if $y < -\frac{b_o}{\mu}$, the solution is unbounded as $t \to \infty$. The trajectories for the case $b_o \neq 0$ can be obtained from those of fig. 1 by replacing y by $y + \frac{b_o}{\mu}$, as is seen from (19). We see that, if $b_o > 0$, the line $y_o = -\frac{b_o}{\mu}$ is a separatrix (corresponding to $A = 0$), separating those trajectories which converge to an equilibrium point on the x axis from those which diverge to infinity.

<u>Case 3</u> $\mu < 0$, $c_1 = 0$.

Using a scaling factor $\alpha = -1$, we can reduce this to Case 2. The trajectories in the phase plane are given by (19), the point $(\mu x, y)$

in figure 1 being transformed into the point $(-|\mu|x, -y)$. We see that, if $\mu < 0$, the solution is bounded as $t \to \infty$ provided that $b_0 > 0$ and $y_0 < -\frac{b_0}{\mu}$.

1.3 Systems with damping and positive stiffness

Case 4 $\quad \mu = 0,\ c_1 > 0.$

$$\ddot{x} + b_0\dot{x} + c_1 x = 0. \tag{23}$$

Equation (23) is a linear differential equation. The general solution and the trajectories in the phase plane were discussed in the previous report (Babister, 1973).

Case 5 $\quad \mu > 0,\ c_1 > 0.$

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x = 0. \tag{24}$$

We first consider the equation

$$\ddot{x} + \mu\dot{x}^2 + c_1 x = 0 \tag{25}$$

This is of the form

$$\ddot{x} + \mu\dot{x}^2 + g(x) = 0 \tag{26}$$

On putting $y = \dot{x}$, (26) can be reduced to the first order equation

$$y\frac{dy}{dx} + \mu y^2 + g(x) = 0$$

which can be integrated to give

$$y^2 e^{2\mu x} = A - 2\int g(x)e^{2\mu x}\,dx \tag{27}$$

where A is a constant.

Equation (25) has the general first integral

$$y^2 = -\frac{c_1 x}{\mu} + \frac{c_1}{2\mu^2} + Ae^{-2\mu x} \tag{28}$$

where $y = \dot{x}$.

Phase plane trajectories are given in fig. 2. We see that, in the neighbourhood of O, the trajectories are closed, O being a centre. The parabola PQR (for which A = 0) is a separatrix, separating the closed trajectories $\left(-\frac{c_1}{2\mu^2} < A < 0\right)$ from the open ones $(A > 0)$ which tend to infinity as $t \to \pm\infty$. We note that the diagram is symmetrical w.r.t. the x axis; from (27) we see that the phase-plane diagram for any system of the form (26) is symmetrical w.r.t. the x axis. If, in (26), $g(0) = 0$ and $\frac{g(x)}{x}$ is positive throughout a small region which includes the origin O, then by a simple topological argument it can be shown that O is a centre for (26). This is illustrated below in cases 6, 17, 18, 20 and 21.

From fig.2 we see that, for equation (25), there is a periodic solution for any system having $\mu x_0 < \frac{1}{2}$, $y_0 = 0$ at $t = 0$. The variation of x with t (as determined by analogue computer) is shown in fig. 3 for various values of μx_0. We see that the period varies little with x_0 (for $\mu x_0 < 0.4$). An approximate formula for the period can be found by writing the solution of (25) in the form

$$x = X_0 + \mu X_1 + \mu^2 X_2 + \dots$$

and

$$c_1 = \alpha_0 + \mu\alpha_1 + \mu^2\alpha_2 + \dots$$

μ being considered as the perturbation parameter. This is Poincaré's method. We find that the period is (to the second order in μ)

$$\frac{2\pi}{\sqrt{c_1}}\left(1 + \frac{\mu^2 x_0^2}{6}\right).$$

Fig. 4 shows the trajectories for the system (24) if $b_0 > 0$ (there is then no exact first integral). O is a stable focus and, for much of the phase plane, the trajectories spiral in to 0. This is to be expected since, for this system, from (9), the energy will be decreasing if $b_0 + \mu y > 0$. However, for trajectories such as AB, penetrating well into the region $-y < b_0/\mu$, the non-linear term μy^2 predominates and the trajectories are open and tend to infinity as $t \to \infty$.

The trajectories of the system (24) with $b_o < 0$ can be obtained by employing a scaling factor $\beta = -1$. Thus the point (x,y) in figure 4 is transformed into the point $(x,-y)$; in addition the arrows on the curves should be reversed. We see that the trajectories spiral away from 0; thus, if $b_o < 0$, all the trajectories tend to infinity as $t \to \infty$.

Case 6 $\mu < 0, \quad c_1 > 0.$

Using a scaling factor $\alpha = -1$, we can reduce this to case 5. As above, we find that, if $b_o = 0$, self sustaining oscillations around 0 can occur if $\mu x_o < \frac{1}{2}$, $y_o = 0$, 0 being a centre. If $b_o > 0$, the trajectories spiral in towards 0 unless x_o is large and negative; if $b_o < 0$, all the trajectories tend to infinity as $t \to \infty$.

On differentiating (24) with respect to t and putting $\dot{x} = z$, we obtain

$$\ddot{z} + (b_o + 2\mu z)\dot{z} + c_1 z = 0,$$

an equation which was considered in the previous report.

1.4. Systems with damping and negative stiffness

Case 7 $\mu = 0, \quad c_1 < 0.$

$$\ddot{x} + b_o\dot{x} + c_1 x = 0. \quad (29)$$

Equation (29) is a linear differential equation. The general solution and the trajectories in the phase plane were discussed in the previous report (Babister, 1973).

<u>Case 8</u> $\mu > 0, c_1 < 0.$

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_1 x = 0. \qquad (30)$$

If $b_o = 0$, equation (30) has the general first integral given by (28). Phase plane trajectories are given for system (30) in fig. 5 ($b_o = 0$) and fig. 6 ($b_o > 0$). These figures have much in common with those for the linear system ($\mu = 0$). We see that O is a saddle point.

Trajectories for $b_o < 0$ can be obtained from those given in fig. 6 by employing a scaling factor $\beta = -1$ (as in case 5). For all values of b_o (with $c_1 < 0$), all the trajectories tend to infinity as $t \to \infty$, except for the two trajectories which converge on the origin; these are two separatrices of the singular point O.

<u>Case 9</u> $\mu < 0, c_1 < 0.$

Using a scaling factor $\alpha = -1$, we can reduce this to case 8. All the trajectories tend to infinity as $t \to \infty$ with the exception of two separatrices of the singular point O.

PART 2

Non-linear differential equations of the form

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_1x + c_2x^2 = 0$$

2.1. Introduction

We consider solutions of the equation

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_1x + c_2x^2 = 0, \tag{31}$$

or of the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -b_oy - \mu y^2 - c_1x - c_2x^2 \end{aligned}\right\} \tag{32}$$

where b_o, c_1, c_2 and μ are constants $(c_2 \neq 0)$. The point 0 is an isolated singular point for the system (32); this point is a non-elementary singular point if $c_1 = 0$. As in part 1, we shall consider the trajectories in the phase plane; in particular we shall determine whether or not there exist trajectories tending towards 0.

2.2 Systems with square-law stiffness $(c_1 = 0, \ c_2 > 0)$

Case 10 $\mu = 0, \ c_1 = 0, \ c_2 > 0.$

$$\ddot{x} + b_o\dot{x} + c_2x^2 = 0 \tag{33}$$

This case was considered in the previous report (Babister, 1973). It was shown that, if $b_o \leqslant 0$, the trajectories (in general) tend to infinity as t increases. If $b_o > 0$, trajectories with $x_o = 0$ and $0 < c_2y_o/b_o^3 < 1.5$ turn inwards to the origin in a similar manner to that for a stable node. All the other trajectories tend to infinity as t increases.

Case 11 $\mu > 0, \ c_1 = 0, \ c_2 > 0.$

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_2x^2 = 0 \tag{34}$$

We first consider the equation

$$\ddot{x} + \mu\dot{x}^2 + c_2x^2 = 0. \tag{35}$$

From (27) we see that (35) has the general first integral

$$y^2 = -\frac{c_2x^2}{\mu} + \frac{c_2x}{\mu^2} - \frac{c_2}{2\mu^3} + A\,e^{-2\mu x} \tag{36}$$

where $y = \dot{x}$ and A is a constant.

Phase plane trajectories are given in fig. 7. The diagram is symmetric w.r.t. the x axis. There is one trajectory ($A = c_2/2\mu^3$, $y > 0$) which terminates at 0; all other trajectories tend to infinity as t increases.

Trajectories for the system (34) for $b_o > 0$ are shown in fig. 8 (for $b_o\sqrt{\mu/c_2} = 1$). There is no general first integral. As in case 10, the slopes λ of trajectories passing through 0 are given by $\lambda = 0$ or $-b_o$. We see that most of the trajectories tend to infinity as t increases, only those with sufficiently small y_o (at $x_o = 0$) turning in towards the origin.

Trajectories for $b_o < 0$ can be obtained by using a scaling factor $\beta = -1$. Thus the point (x,y) in figure 8 is transformed into the point (x,-y); in addition the arrows on the curves should be reversed (t being changed to -t). All the trajectories tend to infinity as t increases (except for one trajectory which terminates at 0). If $b_o < 0$ there are some trajectories which commence at 0.

<u>Case 12</u> $\mu < 0$, $c_1 = 0$, $c_2 > 0$.

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_2x^2 = 0. \tag{37}$$

If $b_o = 0$, (37) has the general first integral given by (36). Phase plane trajectories are given for system (37) in fig. 9 ($b_o = 0$). From fig. 9 and eq. (36) we see that the trajectories can be separated into two groups; if $A \geqslant 0$, trajectories do not intersect the x axis (the trajectory $A = 0$ is a rectangular hyperbola PQ, P'Q'), whereas if $A < 0$ trajectories start in the second quadrant and tend to infinity in the third quadrant. One trajectory $(A = -c_2/2\mu^3)$ terminates at 0 (a non-elementary singular point); all other trajectories tend to infinity as t increases.

Trajectories for the system (37) for $b_o > 0$ are shown in fig. 10 (for $b_o\sqrt{-\mu/c_2} = 1$). We see that for this value of b_o there are two straight line trajectories converging on 0, given by

$$y = -b_o x.$$

This corresponds to the particular integral

$$\dot{x} = -b_o x \tag{38}$$

of (37) if $\mu b_o^2/c_2 = -1$. Putting $u = \dot{x} + b_o x$, we see that (37) becomes

$$\dot{u} + \mu u(\dot{x} - b_o x) = 0. \tag{39}$$

From fig. 10 we see that no trajectory crosses the line AOB, $y = -b_o x$. Trajectories to the right of this line (in the first, second and fourth quadrants) turn in towards 0, having a common tangent along Ox at 0. As can be seen from (39), if $\mu < 0$ and $b_o x > \dot{x}$, trajectories on either side of AO approach this line as t increases, but do not merge with it. Trajectories to the left of AOB (in the second, third and fourth quadrants) tend to infinity as t increases; one of them OC starts from 0.

Trajectories for $b_o < 0$ are obtained by using a scaling factor $\beta = -1$ (as in case 11). All the trajectories tend to infinity as t increases (except for one trajectory which terminates at 0).

2.3 <u>Systems with Square-law Stiffness</u> $(c_1 = 0, c_2 < 0)$

<u>Case 13</u> $\mu = 0, c_1 = 0, c_2 < 0.$

<u>Case 14</u> $\mu > 0, c_1 = 0, c_2 < 0.$

<u>Case 15</u> $\mu < 0, c_1 = 0, c_2 < 0.$

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_2 x^2 = 0 \tag{40}$$

On putting $x = -X$, (40) becomes

$$\ddot{X} + b_0\dot{X} - \mu\dot{X}^2 - c_2X^2 = 0 \tag{41}$$

which is of the same form as (33), (34) or (37), depending on the sign of μ. Thus the integral curves can be found by applying a scaling factor $\alpha = -1$ to those of cases 10, 12 and 11. The point (x,y) is transformed into the point (-x,-y). The behaviour of trajectories in the neighbourhood of the origin is therefore the same as that discussed in para. 2.2. We see that, in general, the trajectories tend to infinity as t increases, except that (i) if $b_0 > 0$, some trajectories close to the origin (in the second, third and fourth quadrants) terminate at O and (ii) there is always one arm of the separatrix which terminates at O. As in para. 2.2, linear trajectories occur for $\mu b_0^2/c_2 = -1$.

2.4. <u>Systems with quadratic stiffness</u> ($c_1 > 0$, $c_2 > 0$)

As pointed out in the general introduction, the system

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1x + c_2x^2 = 0 \tag{42}$$

has equilibrium points at both $x = 0$ and $x = -c_1/c_2$.
Thus, on putting $x = z - c_1/c_2$, we obtain the system

$$\ddot{z} + b_0\dot{z} + \mu\dot{z}^2 - c_1z + c_2z^2 = 0. \tag{43}$$

The phase plane thus has two singular points, but they are both elementary singular points. These singularities coalesce if $c_1 = 0$, giving rise to a non-elementary singular point, as stated in para. 2.1.

<u>Case 16</u> $\mu = 0$, $c_1 > 0$, $c_2 > 0$.

$$\ddot{x} + b_0\dot{x} + c_1x + c_2x^2 = 0. \tag{44}$$

This case was considered in the previous paper (Babister, 1973). It was shown that O is a centre (if $b_0 = 0$), a stable focus ($b_0 > 0$) and an unstable focus ($b_0 < 0$); the other singularity is a saddle point. The nature of the trajectories in the immediate neighbourhood of these two points is determined (in this case) by the linear terms in eq. (42) and (43). Trajectories which always remain sufficiently far from 0 tend to infinity for all values of b_0 as t increases.

Case 17 $\mu > 0, c_1 > 0, c_2 > 0$.

$$\ddot{x} + b_0\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0 \tag{45}$$

This is the most general case of (31) considered so far. As in case 16, the system (45) has two elementary singular points, at 0 and at $x = -c_1/c_2$, the latter point always being a saddle point in the phase plane for (45) with c_1 positive and $c_2 \neq 0$.

We first consider the equation

$$\ddot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0 \tag{46}$$

From (27), we see that (46) has the general first integral

$$y^2 = -\frac{c_2 x^2}{\mu} + \left(\frac{c_2}{\mu^2} - \frac{c_1}{\mu}\right) x - \left(\frac{c_2}{2\mu^3} - \frac{c_1}{2\mu^2}\right) + A\,e^{-2\mu x} \tag{47}$$

where $y = \dot{x}$ and A is a constant.

Phase plane trajectories are given in fig.11. We see that, with $b_0 = 0$, no trajectory tends to 0. 0 is a centre and P, the point $(-c_1/c_2, 0)$, is a saddle point (as in case 16); in fact the term in μ (> 0) makes comparatively little difference to the phase plane diagram (apart from a question of scale). The curve APBPC (the separatrix) divides the phase plane into three regions, all the trajectories going to infinity unless they are within the region PBP. On the arms of the separatrix, one trajectory AP starts from infinity and terminates at the singular point P, another PBP both starts and ends at P, and a third PC starts from P and goes to infinity as t increases.

Figure 12 shows the trajectories for (45) for $b_0/\sqrt{c_1} = 1$ (with $\mu c_1/c_2 = 1$). The origin is a stable focus (or stable node) for positive values of b_0 and an unstable one for negative b_0 (as in case 16). There is a saddle point at the point $(-c_1/c_2, 0)$.

Trajectories for $b_o/\sqrt{c_1} = -1$ can be obtained immediately from fig. 12 by applying a scaling factor $\beta = -1$.

Case 18 $\mu < 0,\ c_1 > 0,\ c_2 > 0$.

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0. \qquad (48)$$

If $b_o = 0$, (48) has the general first integral given by (47). Phase plane trajectories are given for the system (48) in fig. 13 ($b_o = 0$, $\mu c_1/c_2 = -1$). As in case 17, O is a centre and P, the point $(-c_1/c_2, 0)$, is a saddle point. We see that there are four straight line trajectories, two of which (BP, CP) converge on the saddle point; the other two diverge from it. These are the separatrices, given by

$$y = \pm\sqrt{-\frac{c_2}{\mu}}\left(x + \frac{c_1}{c_2}\right).$$

This corresponds to the particular integral

$$c_2\dot{x}^2 = c_1(c_2 x + c_1)^2 \qquad (49)$$

of (48) if $b_o = 0$ and $\mu c_1/c_2 = -1$. We see that the separatrices divide the phase plane into four regions, all the trajectories going to infinity unless they are within the region to the right of PA and PB. In that region there are cyclic trajectories, enclosing the origin. If $b_o > 0$, O is a stable focus (or stable node) and P is a saddle point. Figure 14 shows the trajectories for (48) for $b_o/\sqrt{c_1} = 1$ with $\mu c_1/c_2 = -1$. The trajectories are similar to those shown in fig. 12. Trajectories for $b_o/\sqrt{c_1} = -1$ can be obtained from fig. 14 by applying a scaling factor $\beta = -1$; O is an unstable focus (or unstable node) for $b_o < 0$.

2.5. Systems with quadratic stiffness ($c_1 > 0$, $c_2 < 0$)

Case 19 $\mu = 0,\ c_1 > 0,\ c_2 < 0$.

Case 20 $\mu > 0,\ c_1 > 0,\ c_2 < 0$.

Case 21 $\mu < 0,\ c_1 > 0,\ c_2 < 0$.

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0 \qquad (50)$$

On putting $x = -X$, (50) becomes

$$\ddot{X} + b_o\dot{X} - \mu\dot{X}^2 + c_1X - c_2X^2 = 0 \tag{51}$$

which is of the same form as (44), (45) or (48) (depending on the sign of μ). Thus the integral curves can be found by applying a scaling factor $\alpha = -1$ to those of cases 16, 18 and 17. The point (x,y) is transformed into the point $(-x,-y)$. We find that 0 is a centre if $b_o = 0$, a stable focus if $b_o > 0$ and an unstable focus if $b_o < 0$. The singularity at $(-c_1/c_2, 0)$ is a saddle point.

2.6. <u>Systems with quadratic stiffness</u> $(c_1 < 0)$

<u>Case 22</u> $\mu = 0$, $c_1 < 0$, $c_2 > 0$.

<u>Case 23</u> $\mu > 0$, $c_1 < 0$, $c_2 > 0$.

<u>Case 24</u> $\mu < 0$, $c_1 < 0$, $c_2 > 0$.

<u>Case 25</u> $\mu = 0$, $c_1 < 0$, $c_2 < 0$.

<u>Case 26</u> $\mu > 0$, $c_1 < 0$, $c_2 < 0$.

<u>Case 27</u> $\mu < 0$, $c_1 < 0$, $c_2 < 0$.

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_1x + c_2x^2 = 0. \quad (c_1 < 0) \tag{52}$$

In (52) put $x = z - c_1/c_2$. Then

$$\ddot{z} + b_o\dot{z} + \mu\dot{z}^2 - c_1z + c_2z^2 = 0. \quad (c_1 < 0) \tag{53}$$

We see that the effect of this transformation is to change the sign of the coefficient of x while leaving all the other coefficients unaltered. Thus the nature of solutions for cases 22 - 27 can be determined from those for cases 16 - 21, allowing for the displaced position of the origin of the new phase plane diagrams. We note, too, that cases 25, 26 and 27 can be related to cases 22, 24 and 23 respectively by applying a scaling factor $\alpha = -1$.

We find that, for cases 22 - 27, 0 is a saddle point, corresponding to a position of unstable equilibrium, as would be expected from a consideration of the linear terms in (52). For these cases, the other singular point P $(-c_1/c_2, 0)$ is a centre if $b_o = 0$, a stable focus if $b_o > 0$ and an unstable focus if $b_o < 0$. Thus periodic solutions can occur in the neighbourhood of the point P if $b_o = 0$.

General Conclusions

In this paper and in the previous one (Babister 1973), we have been dealing with free autonomous systems, given by

$$\ddot{x} + b_o\dot{x} + \mu\dot{x}^2 + c_1 x + c_2 x^2 = 0 \tag{54}$$

and, in the previous paper,

$$\ddot{x} + b_o\dot{x} + b_1 x\dot{x} + c_1 x + c_2 x^2 = 0. \tag{55}$$

As shown by Willems (1970) the null solution $(x = 0, \dot{x} = 0)$ of these systems is asymptotically stable if all the roots of the characteristic equation

$$\lambda^2 + b_o\lambda + c_1 = 0$$

have negative real parts, that is, if b_o and c_1 are both positive (as in figures 4, 12 and 14). However, as shown in these figures, such systems are not globally asymptotically stable.

Periodic solutions of (54) occur if $b_o = 0$ and $c_1 > 0$, for all values of μ and c_2 (for sufficiently small amplitudes in x). This is in contrast to the system (55), where such solutions only occur if both $b_o = 0$ and $b_1 c_2 = 0$ (with $c_1 > 0$).

No limit cycles occur with systems (54) and (55). It is readily seen from an examination of the figures that, in general, divergent motion will occur for large initial displacements from an equilibrium position. It will be noted that, for most of the graphs presented in this report, both the ordinate and abscissa are proportional to μ; it follows that any region of stability in the $(x,\dot{x})$ plane can be enlarged by decreasing the value of $|\mu|$. Similarly, decreasing the value of $|c_2|$ moves the separatrices associated with the point $(-c_1/c_2, 0)$ further from the null point. Thus, for both (54) and (55), decreasing the relative importance of the non-linear parameters is to be recommended to ensure a greater domain of attraction for systems perturbed from the equilibrium state.

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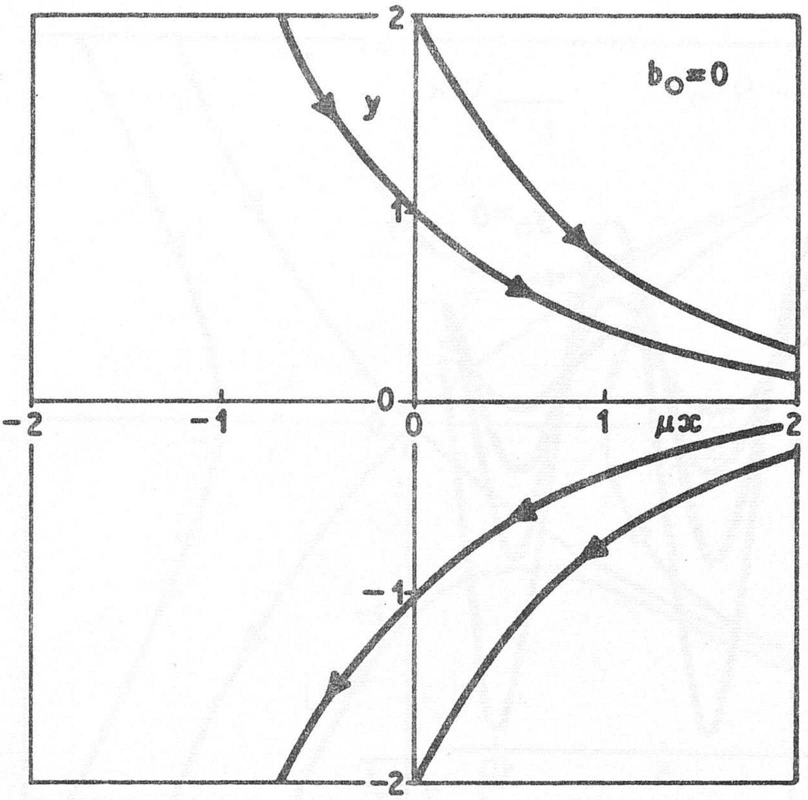

FIG. 1. TRAJECTORIES $\mu > 0, c_1 = 0, c_2 = 0$

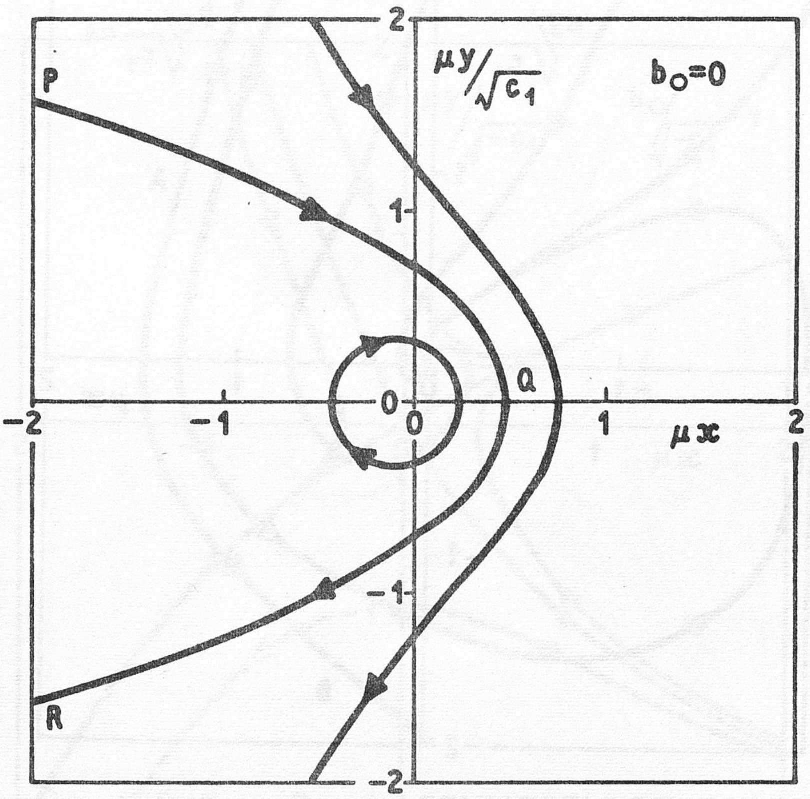

FIG. 2. TRAJECTORIES $\mu > 0, c_1 > 0, c_2 = 0$

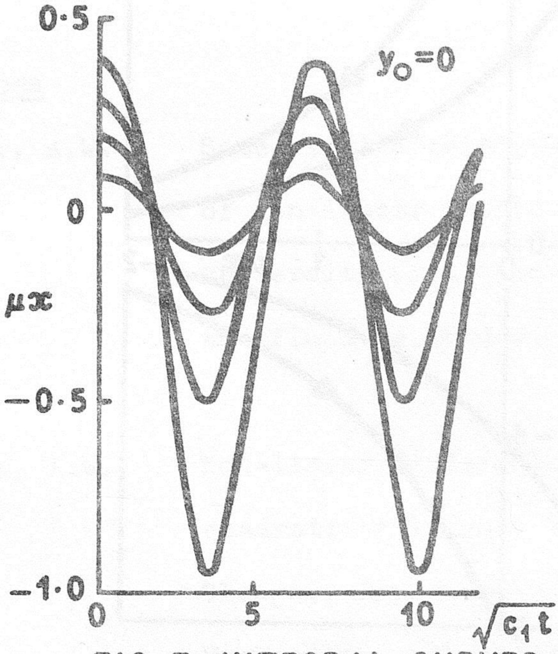

FIG. 3. INTEGRAL CURVES
$b_0=0,\ c_1>0,\ c_2=0$

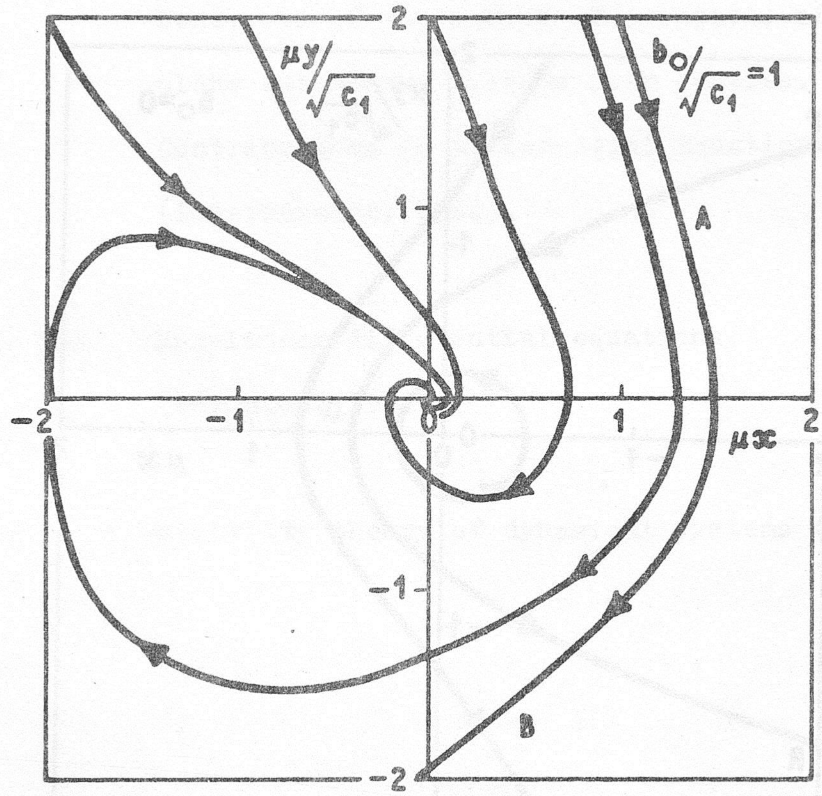

FIG. 4. TRAJECTORIES $\mu>0,c_1>0,c_2=0$

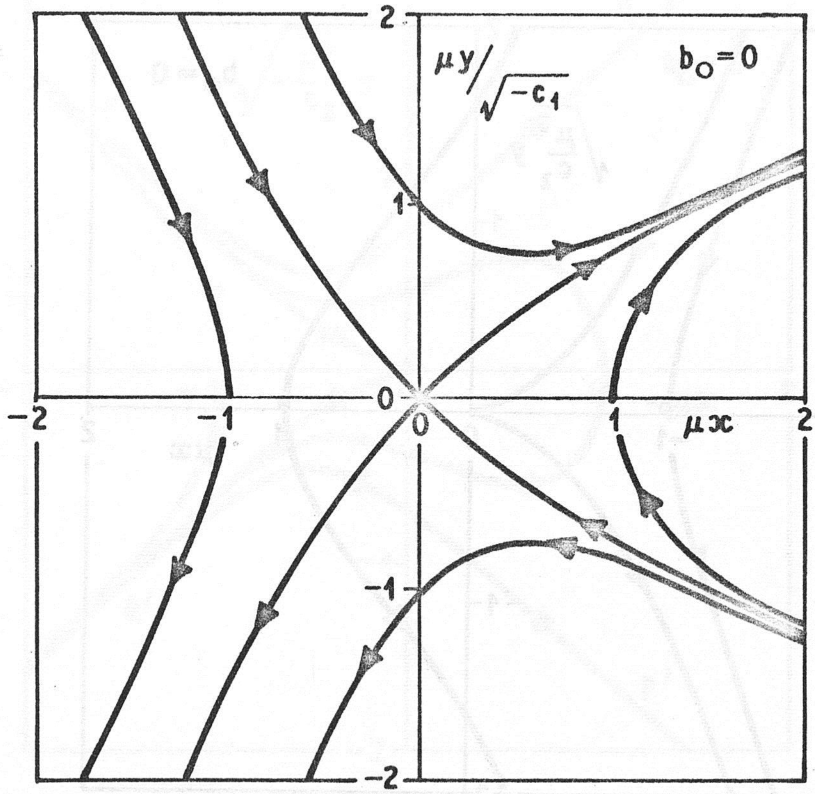

FIG. 5. TRAJECTORIES $\mu > 0, c_1 < 0, c_2 = 0$

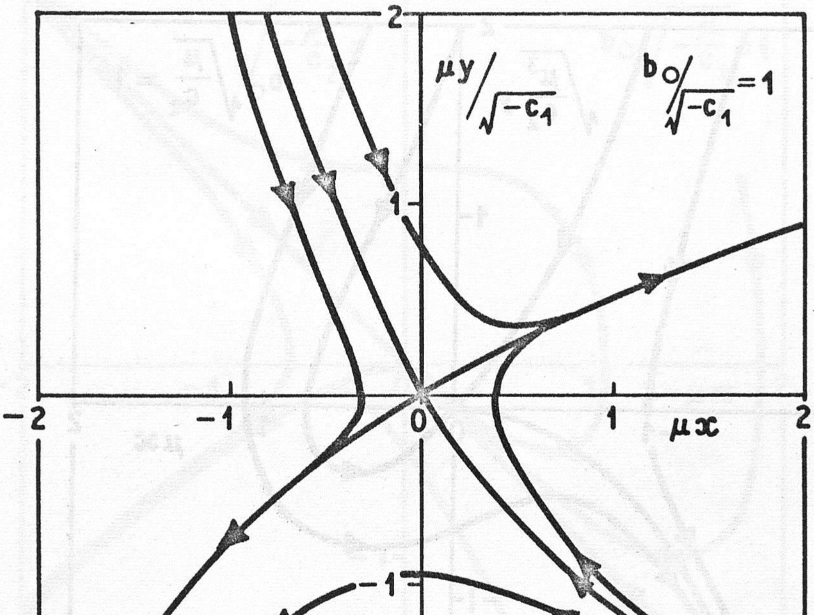

FIG. 6. TRAJECTORIES $\mu > 0, c_1 < 0, c_2 = 0$

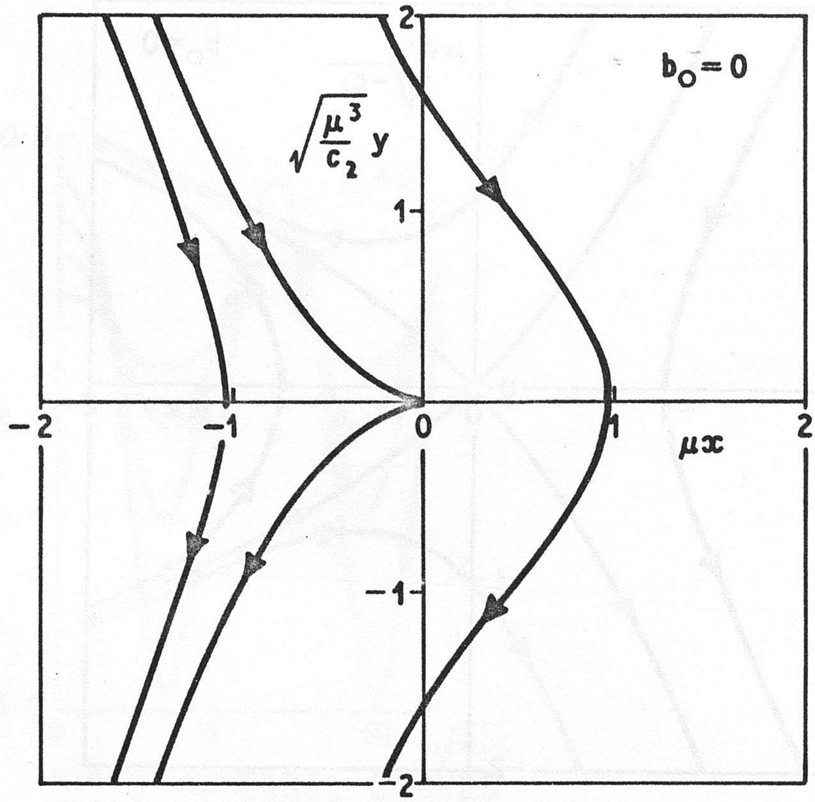

FIG. 7. TRAJECTORIES $\mu > 0, c_1 = 0, c_2 > 0$

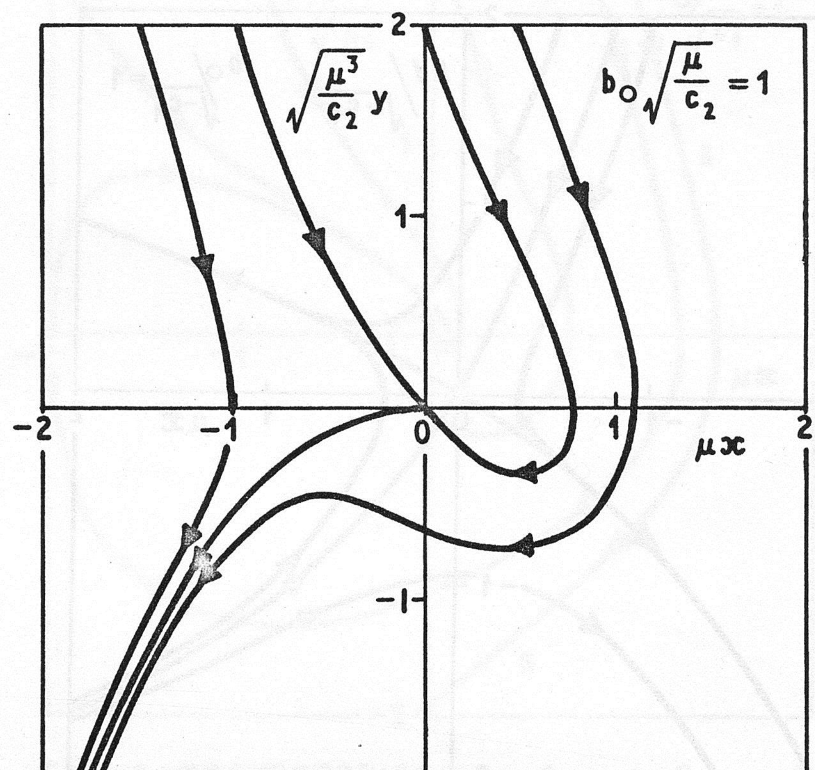

FIG. 8. TRAJECTORIES $\mu > 0, c_1 = 0, c_2 > 0$

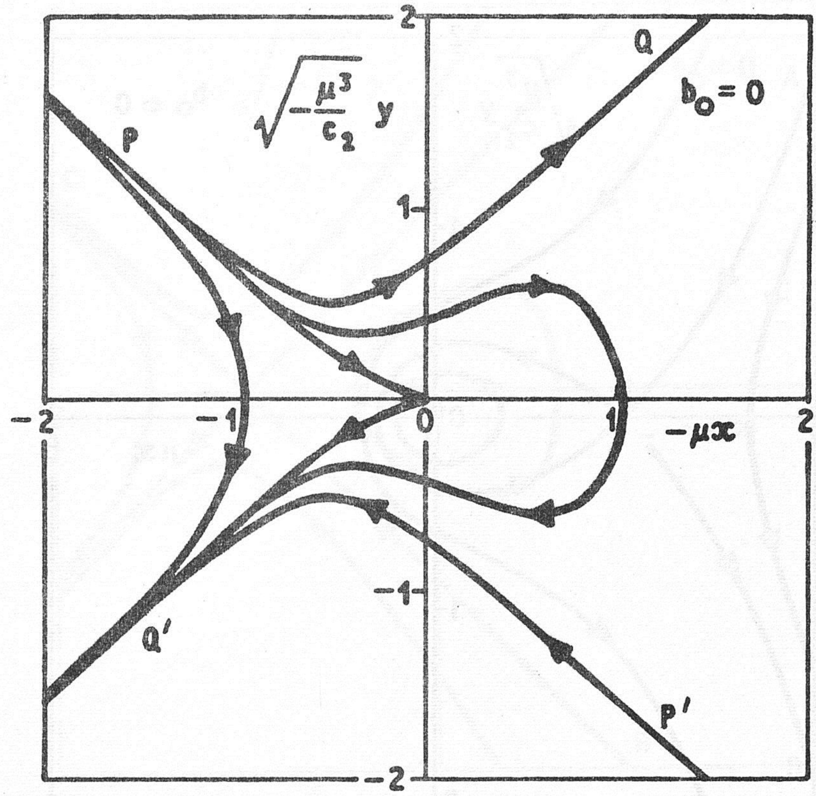

FIG. 9. TRAJECTORIES $\mu < 0, c_1 = 0, c_2 > 0$

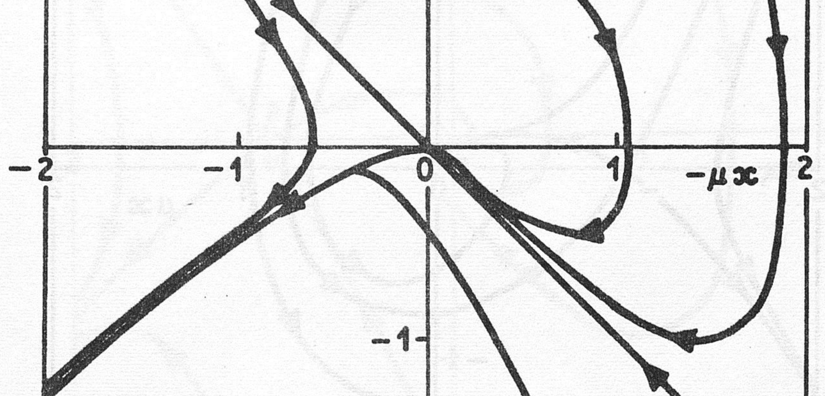

FIG. 10. TRAJECTORIES $\mu < 0, c_1 = 0, c_2 > 0$

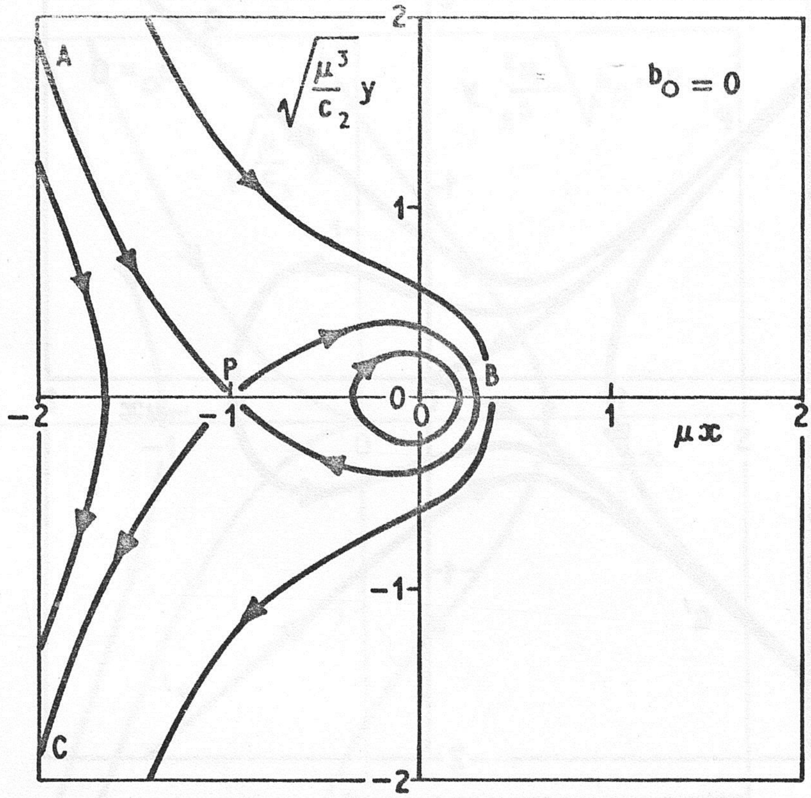

FIG. 11. TRAJECTORIES $\mu > 0, c_1 \geq 0, c_2 > 0$
$(\mu c_1/c_2 = 1)$

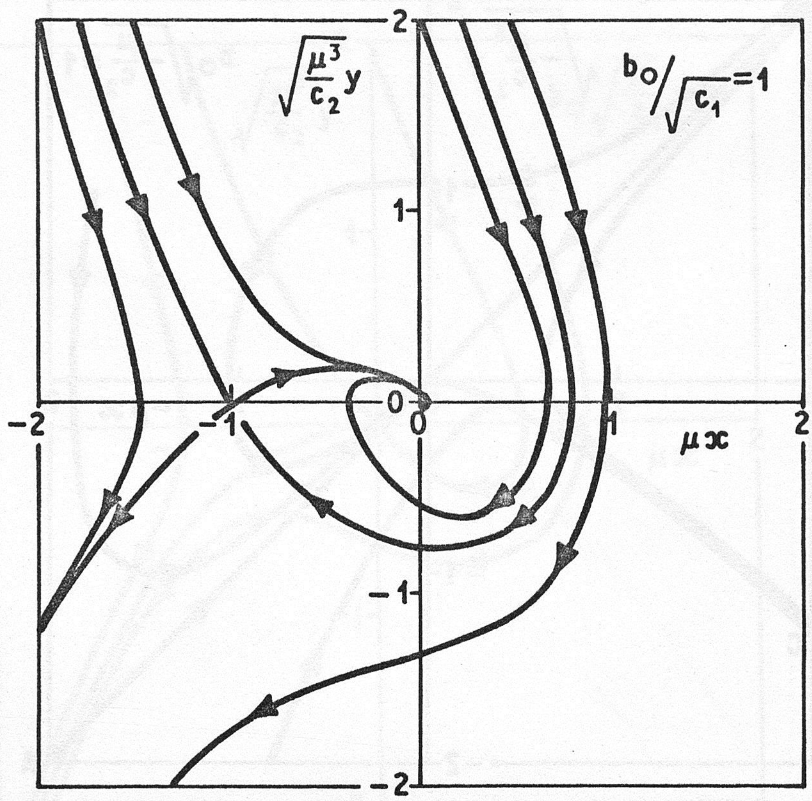

FIG. 12. TRAJECTORIES $\mu > 0, c_1 \geq 0, c_2 > 0$
$(\mu c_1/c_2 = 1)$

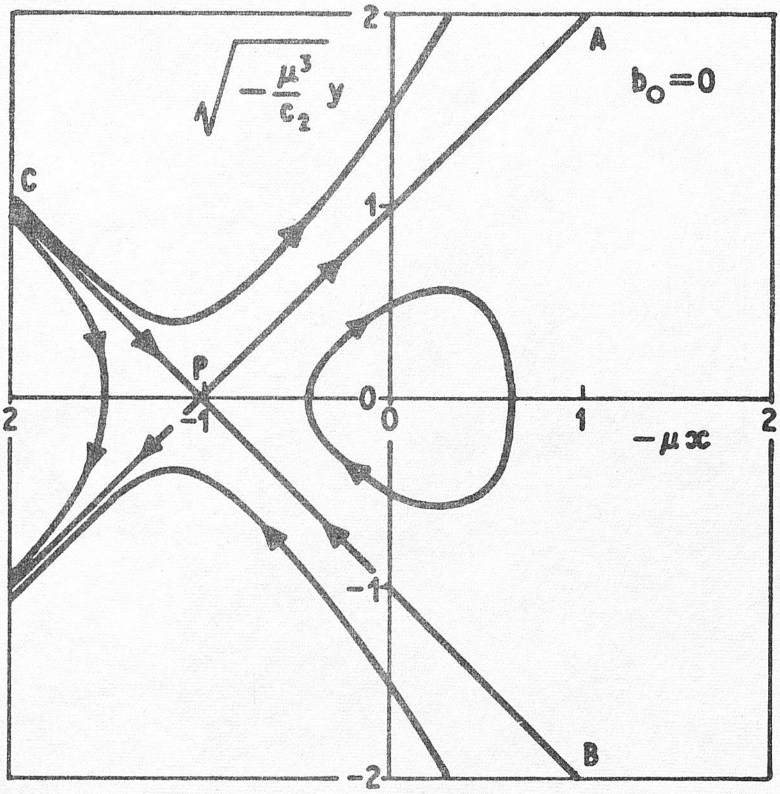

FIG. 13. TRAJECTORIES $\mu<0, c_1>0, c_2>0$
$(\mu c_1/c_2=-1)$

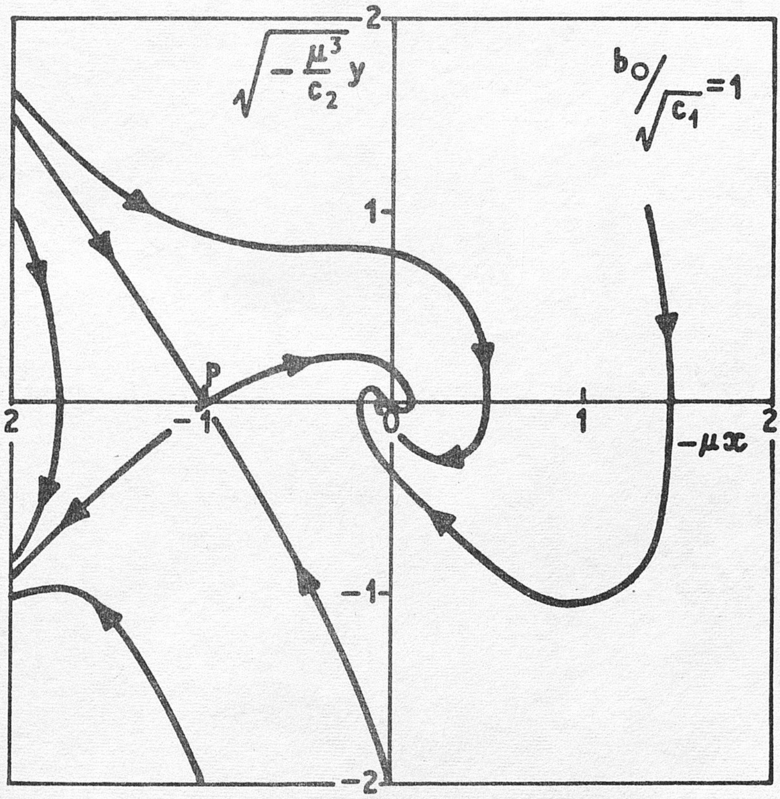

FIG. 14. TRAJECTORIES $\mu<0, c_1>0, c_2>0$
$(\mu c_1/c_2=-1)$